An Investigation of Combinatorial Game Theory and an Analysis of Nim with Pass

An Honors Thesis (HONRS 499)

by

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Abstract

This paper acts as an overview of the basics of combinatorial game theory followed by an in-depth analysis of the combinatorial game Nim with Pass. In doing so, it also provides a discussion of perfect play in traditional Nim as well as a description of P and N-positions. In terms of the analysis of Nim with Pass, it uses a graphical approach which charts P-positions and N-positions of small piles in order to reveal P-positions of larger ones. Although Nim with Pass is not solved, an algorithm to find the position of any game state is offered.

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Author's Statement

My college career has prepared me well for my post-collegiate life—I have been imbued with the knowledge that will shape my financial success in life. However, this thesis is, in fact, a rejection of that knowledge and an embrace of knowledge that is useful only in terms of its aesthetics—knowledge of simple games that few play and provide no actual benefit beyond the pleasure they provide to the players. In simple terms, this is a paper for those people, like me, who find simple joy in playing games rather than complex mathematical analysis of strategy. I take a light-hearted approach to analyzing the strategy of a couple of simple games that anybody can play. There should be little that a layperson in the field will not understand because, when I started, I was also a layperson in the field.

An Introduction to Combinatorial Game theory

Combinatorial game theory[1] refers to what are commonly referred to as "games of no chance." This is because they involve no dice, cards, spins, or any other element of randomness inherent in many games. Common examples are chess, checkers, go, and even tic-tac-toe. In addition to the lack of chance involved, there are a few other requirements that a combinatorial game must adhere to:

- Perfect information—This goes hand in hand with the idea that there is no chance. There are, for instance, no hands of cards that one player is aware of that the other is not as in poker, or cribbage. In fact, there is no information or move that one player is aware of that the other is not.
- Two players—Combinatorial games are strictly played by two players. Having more than two players would require strategy of traditional game theory. Players would need to worry about a diplomacy of sorts as multiple players could "gang up" on other players. This "ganging up" would require strategies which would break the rule of perfect information, as there could be one player unaware of another's alliance.
- Sequential play—Turns are taken one at a time and alternatingly between the two players. Simultaneous turns again would be a breach of the rule of perfect information, as players would implement strategies of traditional game theory—each player basing their move on how they expect their opponent to move.
- Win conditions—Combinatorial games require a condition or set of conditions which will eventually be reached in gameplay that will define a "winner."

Although it is not explicitly stated in the above requirements, there is another rule that combinatorial games adhere to due to the nature of the already defined rules; from any point in the game, one player is able to force a win or at least draw the game no matter what, given perfect play. Obviously, this idea is central to the analysis of combinatorial games.

N-Positions and P-Positions

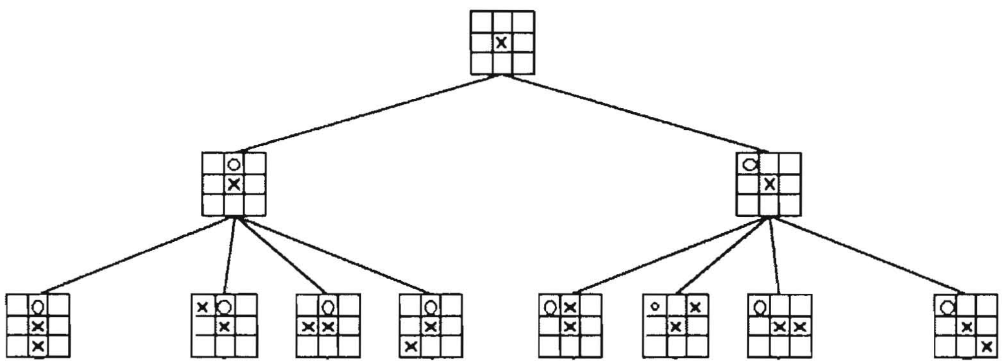
Every game state in a combinatorial game can be classified as one of two kinds of positions. A game state in which the next player to move can force a win is called an N-position; a game state in which the previous player can force a win is called a P-position. P-positions are generally more important in analysis because they represent positions in which it doesn't matter what the next move is—the previous player can still force a win.

[1] As opposed to traditional game theory—the study of games in which an individual's strategy depends upon what he believes his opponents' strategies will be.

This seems strange at first glance—if one player could always win in a combinatorial game, why is it that some players are better than others? The answer is that many combinatorial games are so complex—chess or go, for example—that not even the most powerful computers can deduce the perfect play in most situations.

Therefore, rather than consider the more complicated combinatorial games, consider simple tic-tac-toe. Most children play it at some point, but generally give up playing when they realize that, given perfect play on both parties, the game will always end as a draw. In tic-tac-toe, perfect play is not too hard to figure out—(with X going first) X's best shot is generally to take a corner position, and O must take the middle or will lose, given perfect play on X's part. However, a surprising amount of difficulty is involved in actually showing that this is the case. The obvious way to completely show the perfect series of play is with analyses known as game trees, which, even in tic-tac-toe, can become unmanageable very quickly.

Game trees are a simple enough concept; they are simply a graphical representation of the possible outcomes of a given game. Part of the game tree for tic-tac-toe is shown here:



From this small portion of the tree which has even had duplicate games[2] removed, it is easy to see that the tree will become hard to manage very quickly. The number of branches will reach the thousands by the end of the game. Now it is easy to see why chess has not been fully analyzed—rather than three or four possible moves each turn in a game that will always end on or before turn nine, chess often has upwards of 20 possible moves each turn in a game that has no specific end point. It is clear that chess will have trillions of branches long before the game is over.

[2] A duplicate game refers to games which are essentially the same. For instance, in the above example of tic-tac-toe, O making his first move taking the upper left corner is functionally the same as taking the upper right, bottom left, or bottom right corners.

Solving Combinatorial Games

Along with the idea of game trees comes the idea of "solving" combinatorial games. Because, with a complete game tree, it is possible to tell which player will win, given perfect play, from any point in the game. A game such as this, in which the perfect play is known from any game state whatsoever, is referred to as a "strongly solved game." Tic-tac-toe is one such game—almost anybody can see what the best play is at any point in the game. However, a game can also be "weakly solved." A game is "weakly solved" when the perfect play is known from only the beginning of the game. One example of a game which is only weakly solved is checkers (English draughts). To give an example of the difficulty of solving such large games as checkers, this came after upwards of 50 computers did 10^{14} calculations, working day and night over a period of 18 years (Mullins).

The previous examples have all been games with a finite number of game states. However, there are plenty of games whose game states are not necessarily finite. For instance, the game of Nim is a game that can start with any number of pieces, be it 10 or 10^{100}. In this paper, we will be analyzing a specific type of Nim.

Nim

The game of Nim is a simple combinatorial game in which two players each take turns removing stones from piles. Turns alternate with one player removing at least one stone from a single pile, the winner being the person who removes the last stone[3]. Of course, the game need not be played with stones in piles—it could be pennies in jars, birds in cages, or, more conveniently, simply sets of numbers to be reduced. This is the easiest way to consider a game of Nim:

$\{1, 4, 7\}$

This particular game of Nim would of course represent three piles—one with 1 stone, one with 4 stones, and one with 7 stones. The player who moves next will have 12 possible moves $(1 + 4 + 7)$ which could end up with 12 new game states:

$\{0, 4, 7\}$	$\{1, 3, 7\}$	$\{1, 2, 7\}$	$\{1, 1, 7\}$	$\{1, 0, 7\}$	$\{1, 4, 6\}$
$\{1, 4, 5\}$	$\{1, 4, 4\}$	$\{1, 4, 3\}$	$\{1, 4, 2\}$	$\{1, 4, 1\}$	$(1, 4, 0\}$

Of course, one could imagine infinite games of Nim as there can be as many stones in as many piles as one would want. For instance, this is another example of a game of Nim:

[3] Rules also exist in Nim and most other combinatorial games in which each player's objective is not to take the last move, but instead to force the opponent to take the last move. This style of play is referred to as misère. In fact, traditional Nim is often played as a misère game, although this has little effect on the optimal strategy of traditional Nim.

$\{234, 4878, 1, 10009, 23, 96725, 1120974, 20\}$

This particular game represents eight piles. However, the player who moves next in this game will have 1232864 possible moves. It is clear that simple use of game trees will not suffice in trying to analyze the game of Nim; there is no upper bound to how many branches the game of Nim could have, unless we fix the starting conditions. Taking this idea into consideration makes the seemingly simple game seem impossible to fully analyze or solve. However, Nim is, in fact, a strongly solved game.

Nim-Sums

In order to fully explain how Nim is solved, we must first understand a specific mathematical operation—nim-sums. A nim-sum is a mathematical operation that works differently than regular sums. Rather than adding the two numbers together, we instead convert the summands into a sum of different powers of two, and then remove pairs of matching powers of two, leaving only the powers which were produced an odd number of times. The operator symbol for nim-sums is $\oplus$. For instance,

$$7 \oplus 11 = 12$$

Dissecting this operation, seven can be thought of as $2^0 + 2^1 + 2^2 = 1 + 2 + 4$; eleven can be thought of as $2^0 + 2^1 + 2^3 = 1 + 2+8$. To find $7 \oplus 11$, we add these together, then remove all pairs of matching powers of two, and finally add what is left. For example,

$$7 \oplus 11 \rightarrow (1 + 2 + 4) + (1 + 2 + 8) \rightarrow \cancel{1} + \cancel{2} + 4 + \cancel{1} + \cancel{2} + 8 \rightarrow 4 + 8 = 12$$

As can be seen, a pair of 1's and a pair of 2's were removed, leaving only $4 + 8$. We can also perform this operation in base 2 (binary):

$$\begin{array}{rcr} 7 & = & 0111 \\ \oplus\ 11 & = & 1011 \\ \hline 12 & = & 1100 \end{array}$$

Using the binary method, we simply count the number of 1's in each column, and if it is even, we put a zero in that column; if it is odd, we put a one in that column. Taking this into account, it just so happens that a game's nim-sum is very important as it can show who will win any particular game, given perfect play. It can be shown that when a game's nim-sum is zero the position is a P-position.[4] Given this knowledge, it is now easy to find the optimal move in any game of Nim. For instance, in the previous example of

$\{1, 4, 7\}$

[4] This proof is shown in the appendix.

we can look at the current nim-sum and see whether it is equal to zero. If it is, then there is no move that will put us in a better position; if it is not, we must deduce which move will create a P-position.

$1 \oplus 4 \oplus 7 \rightarrow (1) + (4) + (1 + 2 + 4) \rightarrow \cancel{1} + \cancel{4} + \cancel{1} + 2 + \cancel{4} \rightarrow 2$

Our nim-sum is 2, which is not equal to zero. Therefore, we must deduce what move will make the nim-sum equal to zero. It is simple to see that removing two stones from the pile of seven accomplishes this, leaving the position

$\{1, 4, 5\}$

We can show that this is a P-position by again computing the nim-sum and seeing that it is indeed zero:

$1 \oplus 4 \oplus 5 \rightarrow (1) + (4) + (1 + 4) \rightarrow \cancel{1} + \cancel{4} + \cancel{1} + \cancel{4} \rightarrow 0$

We can do the same thing in a much more impressive manner by using another previous example of

$\{234, 4878, 1, 10009, 23, 96725, 1120974, 20\}$

This time, because of space restrictions, we will perform the operation using the binary method:

234	=	000000000000011101010
4878	=	000000001001100001110
1	=	000000000000000000001
10009	=	000000010011100011001
23	=	000000000000000010111
96725	=	000010111100111010101
1120974	=	100010001101011001110
$\oplus$ 20	=	000000000000000010100
		100000101011111100100

We are left with a nim-sum of 1071076, which is again, not equal to zero. This means that we must deduce which play will leave a nim-sum of zero. Looking at

1120974 = 100010001101011001110

we see that it is the only pile with a 1 in the farthest left binary position, and, therefore, we must remove stones from that pile if we want any hope of reaching a nim-sum of zero. Now, in order to deduce how many stones we must remove, we simply find out what we would like that pile to look like and use that to decide how many to remove. We can see that if the pile instead looks like

$$85290 \quad = \quad 000010100110100101010$$

that we will end up with a nim-sum of zero. Therefore, the winning move is to remove

$$1120974 - 85290 = 1035684$$

stones. Again, we can show that we will be left with a P-position:

234	=	000000000000011101010
4878	=	000000001001100001110
1	=	000000000000000000001
10009	=	000000010011100011001
23	=	000000000000000010111
96725	=	000010111100111010101
1035684	=	000010100110100101010
$\oplus$ 20	=	000000000000000010100
		000000000000000000000

Because our move has created a P-position, the next player will have no move which will result in a P-position, meaning that whichever move the next player makes, we can again create a P-position. We can continue this process until the end of the game.

Now that we have explained the game of Nim, we are free to explore our main focus, a game which is a very subtle variant of Nim—Nim with Pass.

Nim with Pass

Nim with Pass is a game proposed in the most recent update to Richard K. Guy & Richard J. Nowakowski's "Unsolved Problems in Combinatorial Games." In it, David Gale asks for an analysis of the game:

> "Nim with pass. David Gale would like to see an analysis of Nim played with the option of a single pass by either of the players, which may be made at any time up to

the penultimate move. It may not be made at the end of the game. Once a player has passed, the game is as in ordinary Nim. The game ends when all heaps have vanished" (Guy, and Nowakowski).

The game seems quite simple, although upon even the most cursory inspection, it is clear that the method of deducing the correct play in a standard game of Nim is not going to work.

In traditional Nim, we saw that the best strategy is to make moves that leave a nim-sum of zero (P-positions), forcing the opponent's move to always result in a game state with a positive nim-sum (N-positions), and again making the move which creates another P-position. In Nim with Pass, however, players actually need to avoid making moves that result in P-positions in traditional Nim. Any time a game state arises which is a P-position in traditional Nim, the next player will simply take the one-time pass, leaving themselves with the P-position. Realizing this is a good starting point for the analysis.

One of the best ways to go about analyzing a game is to play the game and look for patterns. The following is an example of the P-positions in a game of traditional Nim displayed in a graphical way. This one represents two piles, with the number of stones in one pile along the top and the number of stones in the other pile along the side.

	0	1	2	3	4	5
0	P	N	N	N	N	N
1	N	P	N	N	N	N
2	N	N	P	N	N	N
3	N	N	N	P	N	N
4	N	N	N	N	P	N
5	N	N	N	N	N	P

We can see that a very clear pattern has revealed itself. All of the P-positions are along the diagonal: {0, 0}, {1, 1}, {2, 2}, {3, 3}, etc... This is obvious when considering what we've learned already—the only positions whose nim-sum is zero will be those with equally large piles. There are also a number of other notions that become apparent when looking at this model. First, it should be noted that {0, 0} is a P-position, as this is the position that is reached when a game is over, and the player who went last is obviously the winner for taking the last stone. Additionally we should note that the table is symmetrical about the diagonal—this is obvious when one considers the fact that {1, 4} is equivalent to {4, 1}. Finally, perhaps the most important idea to note is the fact that a move can be thought of as a shift vertically or horizontally in the table. For instance, if a game is at the N-position {3, 5}, the next move will result in a position either above or to the left of {3, 5}. Noting this, we can also conclude that there can only ever be one P-position per row and column; the idea of two P-positions in the same row or column would mean that a P-position would be a single move away from another P-position—a clear logical contradiction.

This graphical analysis is useful, but can be improved by the implementation of a third dimension. First, we can consider the above P/N-position chart not as representing two piles, but instead three piles, the first one of which is empty. Now consider this P/N-position chart with a first pile (displayed as "height") consisting of one stone:

Height = 1

	0	1	2	3	4	5
0	N	P	N	N	N	N
1	P	N	N	N	N	N
2	N	N	N	P	N	N
3	N	N	P	N	N	N
4	N	N	N	N	N	P
5	N	N	N	N	P	N

In this model, we again recognize a rather clear pattern, although one different from the first. There is again only one P-position per row and column, although they are now taking the positions {1, 2n, 2n+1}, n = 0, 1, 2, 3, ...[5] Also note that the position {1, 1, 0}[6] in the above chart is equivalent to the position {1, 1} in the previous chart.[5] Now we extend the analysis to first-pile-values up to 5:

[5] Keep in mind that the above P/N-position chart reflects only piles of five or fewer. The pattern can be shown to continue when looking at the enlarged version of the chart located in the appendix.

[6] For future reference, in this paper, positions will always be referred to by {Height, Horizontal distance, Vertical distance}.

Height = 0

	0	1	2	3	4	5
0	P	N	N	N	N	N
1	N	P	N	N	N	N
2	N	N	P	N	N	N
3	N	N	N	P	N	N
4	N	N	N	N	P	N
5	N	N	N	N	N	P

Height = 1

	0	1	2	3	4	5
0	N	P	N	N	N	N
1	P	N	N	N	N	N
2	N	N	N	P	N	N
3	N	N	P	N	N	N
4	N	N	N	N	N	P
5	N	N	N	N	P	N

Height = 2

	0	1	2	3	4	5
0	N	N	P	N	N	N
1	N	N	N	P	N	N
2	P	N	N	N	N	N
3	N	P	N	N	N	N
4	N	N	N	N	N	N
5	N	N	N	N	N	N

Height = 3

	0	1	2	3	4	5
0	N	N	N	P	N	N
1	N	N	P	N	N	N
2	N	P	N	N	N	N
3	P	N	N	N	N	N
4	N	N	N	N	N	N
5	N	N	N	N	N	N

Height = 4

	0	1	2	3	4	5
0	N	N	N	N	P	N
1	N	N	N	N	N	P
2	N	N	N	N	N	N
3	N	N	N	N	N	N
4	P	N	N	N	N	N
5	N	P	N	N	N	N

Height = 5

	0	1	2	3	4	5
0	N	N	N	N	N	P
1	N	N	N	N	P	N
2	N	N	N	N	N	N
3	N	N	N	N	N	N
4	N	P	N	N	N	N
5	P	N	N	N	N	N

Again, we see clear patterns in our P-positions. Also notice that each separate chart has been labeled "Height = _." This is because these charts can be thought of in the third dimension—one could imagine each P/N-position chart as a separate layer in a larger cube which describes all the games of three or fewer piles of five or fewer stones. As can be seen, the rule of "one P-position per row and column" still holds. However, with the implementation of the third dimension, we have a new discovery—there can also only be a single P-position per column in the third dimension. That is, the presence of the P-position {5, 5, 0} in the "Height = 5" chart means that the positions {4, 5, 0}, {3, 5, 0}, {2, 5, 0}, {1, 5, 0}, and {0, 5, 0} will be N-positions as they are all a single move away from {5, 5, 0}. On second reflection, this might appear reasonable enough when realizing that the "height" in these diagrams is not playing any special role, and is instead representing one of the three piles in the game: thus, the rules governing the "height" are the same as those that apply to the displayed rows and columns.

With the above P/N-position charts, one would easily be able to play perfectly, so long as there are three or fewer piles of five stones or less. However, these are not particularly useful since we already knew how to perfectly play Nim of any size using the nim-sum method. However, these charts become quite useful when analyzing variations on the standard game of Nim, such as Nim with Pass. P/N-position charts for Nim up to H = 15 are located in the appendix. Now, with our procedure more clearly defined, we can move on with the analysis of Nim with Pass.

There is no set of rules or algorithm for going about solving a game besides game trees. Obviously, we cannot create a complete game tree for a game which is boundless. Knowing this, the best place to start is in simply playing a few games, finding P-positions, and constructing some P/N-position charts.

Construction of P/N-position Charts for Nim with Pass

In order to make P/N-position charts for Nim with Pass, the best idea is quite possibly to remember that any P-position in traditional Nim will be an N-position in Nim with Pass (all P/N-position charts and other analysis of Nim with Pass assume that the pass has not yet been taken—otherwise, it would be exactly the same as traditional Nim). This is because any player who would be passed a P-position in Nim with Pass can simply take the pass, leaving themselves in the winning position (with the exception of {0, 0, 0} as that still denotes a finished game and must always denote a P-position). Additionally, as a rule, we can conclude that a position is a P-position when every possible move leads to an N-position. Similarly, we can also conclude that a position is an N-position when one move leads to a P-position. Let's implement this in our Nim with Pass P/N-position charts as our first step:

Height = 0

	0	1	2	3	4	5
0	P	N	N	N	N	N
1	N	N				
2	N		N			
3	N			N		
4	N				N	
5	N					N

Height = 3

	0	1	2	3	4	5
0	N			N		
1			N			
2		N				
3	N					
4						
5						

Height = 1

	0	1	2	3	4	5
0	N	N				
1	N					
2				N		
3			N			
4						N
5					N	

Height = 4

	0	1	2	3	4	5
0	N				N	
1						N
2						
3						
4	N					
5		N				

Height = 2

	0	1	2	3	4	5
0	N		N			
1				N		
2	N					
3		N				
4						
5						

Height = 5

	0	1	2	3	4	5
0	N					N
1					N	
2						
3						
4		N				
5	N					

Notice that the P-position {0, 0, 0} reveals that all positions one move away from it are N-positions. With this, we have a good start, but now we need to do some real analysis. The only way to do this is to create a few simple game trees. Let's start with an easy one:

{0, 2, 1}

The next player has four possible moves—take two stones from the second pile, take a single stone from the second pile, take the last stone in the third pile, or take the Pass. These moves leave four new positions (Positions in which the pass has been taken are denoted by $[X_1, X_2, X_3]$ rather than $\{X_1, X_2, X_3\}$):

{0, 0, 1}, {0, 1, 1}, {0, 2, 0}, [0, 2, 1]

We can see from our Height = 0 chart above that {0, 0, 1}, {0, 1, 1}, and {0, 2, 0} are N-positions. It is also obvious that [0, 1, 0] is an N-position as the next player can do nothing except win. Remembering that any position that has only moves which lead to N-positions is always a P-position, we can conclude that {0, 2, 1} is a P-position (as are the equivalent positions {0, 1, 2}, {2, 0, 1}, {2, 1, 0}, {1, 2, 0}, and {1, 0, 2}). Let's put our findings into our P/N-position charts:

Height = 0

	0	1	2	3	4	5
0	P	N	N	N	N	N
1	N	N	P	N	N	N
2	N	P	N	N	N	N
3	N	N	N	N		
4	N	N	N		N	
5	N	N	N			N

Height = 2

	0	1	2	3	4	5
0	N	P	N	N	N	N
1	P	N	N	N	N	N
2	N	N				
3	N	N				
4	N	N				
5	N	N				

Height = 1

	0	1	2	3	4	5
0	N	N	P	N	N	N
1	N		N			
2	P	N	N	N	N	N
3	N		N			
4	N		N			N
5	N		N		N	

Height = 3

	0	1	2	3	4	5
0	N	N	N	N		
1	N		N			
2	N	N				
3	N					
4						
5						

Height = 4

	0	1	2	3	4	5
0	N	N	N		N	
1	N		N			N
2	N	N				
3						
4	N					
5		N				

Height = 5

	0	1	2	3	4	5
0	N	N	N			N
1	N		N		N	
2	N	N				
3						
4		N				
5	N					

For another example, let's look at the position

{1, 1, 1}

Here, there are really only two moves—take the Pass or take the final stone in any of the three piles (remember the equivalence of {1, 1, 0}, {1, 0, 1}, and {0, 1, 1}). These moves leave us two possible positions:

{0, 1, 1}, [1, 1, 1]

We know from the chart above that {0, 1, 1} is an N-position. We can also see that [1, 1, 1] is an N-position from the traditional Nim example. Therefore, we can conclude that {1, 1, 1} is a P-position. Implementing this new knowledge, we get:

Height = 0

	0	1	2	3	4	5
0	**P**	N	N	N	N	N
1	N	N	**P**	N	N	N
2	N	**P**	N	N	N	N
3	N	N	N	N		
4	N	N	N		N	
5	N	N	N			N

Height = 1

	0	1	2	3	4	5
0	N	N	**P**	N	N	N
1	N	P	N	N	N	N
2	**P**	N	N	N	N	N
3	N	N	N			
4	N	N	N			N
5	N	N	N		N	

Height = 2

	0	1	2	3	4	5
0	N	P	N	N	N	N
1	P	N	N	N	N	N
2	N	N				
3	N	N				
4	N	N				
5	N	N				

Height = 4

	0	1	2	3	4	5
0	N	N	N		N	
1	N	N	N			N
2	N	N				
3						
4	N					
5		N				

Height = 3

	0	1	2	3	4	5
0	N	N	N	N		
1	N	N	N			
2	N	N				
3	N					
4						
5						

Height = 5

	0	1	2	3	4	5
0	N	N	N			N
1	N	N	N		N	
2	N	N				
3						
4		N				
5	N					

It is easy to see how to progress from here: We can simply look at the spaces which are still empty and check whether all of the possible moves lead to N-positions. If so, mark it as a P-position and mark all positions one move away from it as N-positions. In this way, we can complete the P/N-position chart we've been working on:

Height = 0

	0	1	2	3	4	5
0	P	N	N	N	N	N
1	N	N	P	N	N	N
2	N	P	N	N	N	N
3	N	N	N	N	P	N
4	N	N	N	P	N	N
5	N	N	N	N	N	N

Height = 1

	0	1	2	3	4	5
0	N	N	P	N	N	N
1	N	P	N	N	N	N
2	P	N	N	N	N	N
3	N	N	N	P	N	N
4	N	N	N	N	P	N
5	N	N	N	N	N	P

Height = 2

	0	1	2	3	4	5
0	N	P	N	N	N	N
1	P	N	N	N	N	N
2	N	N	P	N	N	N
3	N	N	N	N	N	P
4	N	N	N	N	N	N
5	N	N	N	P	N	N

Height = 4

	0	1	2	3	4	5
0	N	N	N	P	N	N
1	N	N	N	N	P	N
2	N	N	N	N	N	N
3	P	N	N	N	N	N
4	N	P	N	N	N	N
5	N	N	N	N	N	N

Height = 3

	0	1	2	3	4	5
0	N	N	N	N	P	N
1	N	N	N	P	N	N
2	N	N	N	N	N	P
3	N	P	N	N	N	N
4	P	N	N	N	N	N
5	N	N	P	N	N	N

Height = 5

	0	1	2	3	4	5
0	N	N	N	N	N	N
1	N	N	N	N	N	P
2	N	N	N	P	N	N
3	N	N	P	N	N	N
4	N	N	N	N	N	N
5	N	P	N	N	N	N

Here, we again notice some patterns, although they are not quite as obvious as those seen in traditional Nim. In the "Height = 0" chart, we see that P-positions follow a pattern of {0, 2n, 2n – 1}. Observing the chart in the appendix, we can see that this pattern continues forever. Also, in the "Height = 1" chart, we see that aside from {1, 0, 2}, {1, 1, 1}, and {1, 2, 0}, the P-positions follow the pattern of {1, n, n}, n >2. Again observing the chart in the appendix, we see that this pattern also continues forever. However, above Height = 1, we see no patterns which are so easily defined. In fact, the patterns that appear tend to only appear for a short time, and then dissipate. In fact, in the research performed, no pattern that could be expected to continue forever was found on any chart with Height greater than one.

Of course, we need not restrict ourselves to piles of five or fewer stones[7]. Additionally, we need not restrict ourselves to only three dimensions, either—rather than having only "Height = _" charts, we could easily create many more higher-dimensional charts with indications such as "Third dimension = _; Fourth dimension = _; Fifth dimension = _; etc…" We can make these as large as we want, given enough time. More

[7] The appendix shows P/N-position charts up to 3 piles of 15.

importantly, in constructing these, we have actually developed an algorithm for finding all possible P-positions relatively quickly.

The Algorithm

While we have not technically solved Nim with Pass, with the research we have done so far, we can develop an algorithm to find all P-positions in Nim with Pass. The algorithm proceeds as follows:

1. First, mark all positions whose nim-sum is zero (except {0, 0, 0, ...}) as N-positions.
2. Mark {0, 0, 0, ...} as a P-position and mark all positions one move away as N-positions.
3. Find the as yet unknown position $\{X_1, X_2, X_3, \ldots X_n\}$ whose $X_1 + X_2 + X_3 + \ldots + X_n$ is smallest.
4. Check that all positions one move behind this position are indeed N-positions. If so, mark the position as a P-position, and mark all positions one move away as N-positions. If not, mark the position as an N-position.
5. Repeat 3 and 4 as necessary.

This algorithm is obviously rather tedious to compute by hand, and would likely be much easier to use with the aid of a computer. Unfortunately, this paper cannot provide a computer program to implement this algorithm. However, it would appear that the creation of such a program would not be terribly difficult for most competent programmers. In fact, the creation of such a program would likely also make for a rather interesting thesis.

References

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Appendix I: P/N-position Charts for Traditional Nim

Height = 0

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
2	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P

Height = 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
2	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N

Height = 2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
1	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
2	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
6	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
10	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
14	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N

Height = 3

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
1	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
2	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
3	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N

Height = 4

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
1	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
4	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
5	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
6	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
12	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N

Height = 5

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
1	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
3	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
4	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
5	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
6	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
12	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N

Height = 6

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N
1	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N
2	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N
4	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N
6	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**
10	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N
12	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N
14	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N

Height = 7

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N
1	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N
6	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N

Height = 8

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
1	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
2	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
5	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
8	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N

Height = 9

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
1	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
2	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
5	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
8	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N

Height = 10

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
1	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
2	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
5	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
6	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
8	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
10	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
14	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N

Height = 11

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
1	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
2	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
3	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
5	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
8	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N

Height = 12

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N
1	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N
2	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N
3	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**
4	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N
6	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N
8	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N
12	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N

Height = 13

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N
1	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N
2	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**
3	N	N	N	N	N	N	N	N	N	N	N	N	N	N	**P**	N
4	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N
5	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N
6	N	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N
7	N	N	N	N	N	N	N	N	N	N	**P**	N	N	N	N	N
8	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N
11	N	N	N	N	N	N	**P**	N	N	N	N	N	N	N	N	N
12	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	**P**	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 14

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
1	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
2	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
3	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
4	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
6	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
8	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
10	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
12	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
14	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 15

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
1	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
2	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
3	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
6	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Appendix II: P/N-position Charts for Nim with Pass

Height = 0

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
2	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 1

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
1	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
2	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P

Height = 2

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
2	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
5	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
7	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N

Height = 3

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
1	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
3	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
6	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
12	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 4

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
1	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
3	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
6	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
7	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
13	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N

Height = 5

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
1	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
2	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
5	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
6	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
8	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
12	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
14	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N

Height = 6

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
1	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
2	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
3	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
5	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
6	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
15	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N

Height = 7

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
1	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
2	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
4	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
6	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
7	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N

Height = 8

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
1	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
2	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
4	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
5	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
6	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
7	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 9

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
1	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
2	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
3	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N
4	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
5	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
6	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
8	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
9	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 10

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
1	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
2	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
5	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N

Height = 11

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
1	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
2	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
5	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
11	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N

Height = 12

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
1	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
2	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
5	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
11	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
12	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
14	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
15	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N

Height = 13

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
1	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
2	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
3	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
6	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
11	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
13	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N

Height = 14

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
1	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
2	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
3	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
5	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
6	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
11	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
13	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
14	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N

Height = 15

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
1	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P
2	N	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N
3	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
4	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N
5	N	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N
6	N	N	N	N	N	N	N	N	N	N	P	N	N	N	N	N
7	N	N	N	N	N	N	N	N	N	N	N	N	N	P	N	N
8	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
9	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N	N
10	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N
11	N	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N
12	N	N	N	N	P	N	N	N	N	N	N	N	N	N	N	N
13	N	N	N	N	N	N	N	P	N	N	N	N	N	N	N	N
14	N	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N
15	N	P	N	N	N	N	N	N	N	N	N	N	N	N	N	N